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Hyper-Hermitian metrics with symmetry

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Abstract

We show that any four-dimensional hyper-Hermitian manifold admitting a non-trivial tri-holomorphic Killing vector field is locally determined by the solution of a monopole-like equation on a three-dimensional Einstein–Weyl space of a special type. Conversely, any four-dimensional hyper-Hermitian manifold admitting a non-trivial tri-holomorphic Killing vector field arises in this way. © 1998 Elsevier Science B.V.

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1. Introduction

In this article our aim is to study four-dimensional hyper-Hermitian spaces which admit a (complete) Killing vector field preserving all the complex structures, a *tri-holomorphic Killing vector field.* Hyper-Hermitian spaces have attracted interest in the physics literature as the target spaces for σ-models with (4, 0)-supersymmetry, cf. [7] and references therein. We find that, given a tri-holomorphic Killing vector field X, such a space is determined locally by the solution of a monopole-like equation on a three-dimensional Einstein–Weyl space of a special type, this Einstein–Weyl space being identified with the "manifold" of orbits of X with the induced metric in the standard way [11]. Furthermore, every four-dimensional hyper-Hermitian space with tri-holomorphic Killing vector field arises in this way. These special Einstein–Weyl spaces form an interesting class in their own right. We show that the only compact example, besides the Riemannian or the flat case, is the Berger sphere regarded as an Einstein–Weyl space [11].

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In Section 2, we review the theory of hypercomplex and hyper-Hermitian spaces and show how, in four dimensions, the existence of a tri-holomorphic Killing vector field leads to a particular geometry on the "manifold of orbits". In Section 3, we interpret this geometry as an Einstein–Weyl connection subject to an extra condition on the curvature. Conversely, we show that given this condition on the curvature the construction can be reversed, leading to a four-dimensional hyper-Hermitian space.

Four-dimensional hyper-Hermitian spaces admitting a tri-holomorphic Killing vector field have been considered in the physics literature [7,14] but the results obtained in the present work are more complete.

2. Hyper-Hermitian 4-manifolds with symmetry

2.1. Almost-hyper-Hermitian structures in dimension 4

A triple $\{I_1, I_2, I_3\}$ of almost-complex structures such that

$$I_1 \circ I_2 = -I_2 \circ I_1 = I_3, \tag{1}$$

on some (C^∞, real) manifold M, is called an *almost-hypercomplex structure*. If, moreover, the I_i's are orthogonal with respect to some Riemannian metric g, we obtain an *almost-hyper-Hermitian structure*. An almost-hypercomplex, resp. almost-hyper-Hermitian, structure is called *hypercomplex, resp. hyper-Hermitian*, if all I_i's are integrable. A hyper-Hermitian structure is called *hyper-Kähler* if all I_i's are parallel with respect to the Levi-Civita connection D^g of g.

If M is four-dimensional, an almost-hypercomplex structure $\{I_1, I_2, I_3\}$ determines an orientation and a conformal class $[g]$ of Riemannian metrics by decreeing that $\{X, I_1X, I_2X, I_3X\}$ is a direct, conformally orthonormal frame of the tangent bundle TM for any non-vanishing vector field X. Conversely, any Riemannian metric with respect to which $\{I_1, I_2, I_3\}$ is almost-hyper-Hermitian belongs to this conformal class.

We denote by A^+M the (rank 3, real) vector bundle of self-dual, skew-symmetric endomorphisms of TM, endowed with the induced inner product: $(A, B) = -\frac{1}{2}\mathrm{trace}(A \circ B)$. Then, two sections A and B of A^+M are orthogonal if and only if they anticommute, and $A \circ A = -1$ if and only if its square-norm is equal to 2 (we thus obtain a natural identification of the set of *all* positive, $[g]$-orthogonal almost-complex structures on M with the set of sections of the sphere bundle $S_{\sqrt{2}}(A^+M)$, the so-called *twistor space* of $(M, [g])$). Accordingly, an orthogonal frame $\{A_1, A_2, A_3\}$ will be called *orthonormal* if the square-norms of the A_i's are all equal to 2. It will be called *direct* if, moreover, $A_3 = A_1 \circ A_2$. Then, an almost-hyper-Hermitian structure with respect to g is nothing else than a (global) trivialization of A^+M by a direct, orthonormal frame. It follows that *any* four-dimensional, oriented, Riemannian manifold locally admits an almost-hyper-Hermitian structure.

We recall the following well-known fact:

Proposition 1. *Let (M, g) be an oriented, four-dimensional Riemannian manifold. Then, A^+M can be locally trivialized by integrable almost-complex structures $\{I_1, I_2, I_3\}$ if and only if the positive Weyl tensor W^+ vanishes identically, i.e. g is anti-self-dual.*

Proof. The part *if* is a direct consequence of the integrability theorem of Atiyah–Hitchin–Singer for the canonical almost-complex structure on the twistor space $S_{\sqrt{2}}(A^+M)$ [2]. The part *only if* is a direct consequence of the following more precise fact: at each point x of M where W^+ is non-zero, there exist two distinguished pairs $\pm J_1, \pm J_2$ of elements of the fibre A_x^+M (reduced to one pair if W^+ is *degenerate* at x) with the following property: *For any integrable, positive, orthogonal almost-complex structure J in the neighbourhood of x, the value at x of J coincides with one of the elements $\pm J_1, \pm J_2$* [17] or [1] (in spinorial notations, the four elements $\pm J_1, \pm J_2$ correspond to the four roots of W^+ viewed as a section of $\Sigma_+^4 M$). □

Note. Recall that A^+M can be locally trivialized by D^g*-parallel* sections if and only if W^+ *and* the Ricci tensor Ric of g both vanish identically [10].

From now on, (M, g) will denote an oriented, four-dimensional Riemannian manifold, equipped with an almost-hyper-Hermitian structure $\{I_1, I_2, I_3\}$, viewed as a direct, orthonormal (global) frame of A^+M.

We denote by $\{\Omega_1, \Omega_2, \Omega_3\}$ the corresponding *Kähler forms*, i.e. the (self-dual) real 2-forms defined by $\Omega_i(\cdot, \cdot) = g(I_i \cdot, \cdot)$, $i = 1, 2, 3$.

For each I_i we consider the real 1-form θ_i, called the *Lee form* of the almost-Hermitian structure (g, I_i), defined by

$$\theta_i = I_i\, \delta\Omega_i, \tag{2}$$

or, equivalently

$$\mathrm{d}\Omega_i = \theta_i \wedge \Omega_i. \tag{3}$$

Note. The action of I_i on the *cotangent space* T^*M is defined by $(I_i\, \alpha)(\cdot) = -\alpha(I_i\, \cdot)$ for any covector α, so as to be compatible with the Riemannian duality between TM and T^*M.

We shall use the following integrability criterion for almost-complex structures in dimension 4.

Proposition 2.

(a) *I_1 is integrable if and only if $\theta_2 = \theta_3$, and similarly for I_2 and I_3.*
(b) *$\{I_1, I_2, I_3\}$ is hypercomplex if and only if $\theta_1 = \theta_2 = \theta_3$.*

Proof. The part (b) is an obvious consequence of (a). In order to prove (a) and for later use, we introduce the real 1-forms $\{\alpha_1, \alpha_2, \alpha_3\}$ determined by

$$
\begin{aligned}
D^g I_1 &= \tfrac{1}{2} I_3\, \alpha_3 \otimes I_2 - \tfrac{1}{2} I_2\, \alpha_2 \otimes I_3, \\
D^g I_2 &= \tfrac{1}{2} I_1\, \alpha_1 \otimes I_3 - \tfrac{1}{2} I_3\, \alpha_3 \otimes I_1, \\
D^g I_3 &= \tfrac{1}{2} I_2\, \alpha_2 \otimes I_1 - \tfrac{1}{2} I_1\, \alpha_1 \otimes I_2.
\end{aligned} \tag{4}
$$

By considering the trace of the right-hand sides of (4), it is easily checked that the 1-forms $\alpha_1, \alpha_2, \alpha_3$ are related to the Lee forms $\theta_1, \theta_2, \theta_3$ by

$$
\begin{aligned}
\theta_1 &= \tfrac{1}{2}(\alpha_2 + \alpha_3), & \alpha_1 &= -\theta_1 + \theta_2 + \theta_3, \\
\theta_2 &= \tfrac{1}{2}(\alpha_3 + \alpha_1), & \alpha_2 &= +\theta_1 - \theta_2 + \theta_3, \\
\theta_3 &= \tfrac{1}{2}(\alpha_1 + \alpha_2), & \alpha_3 &= +\theta_1 + \theta_2 - \theta_3.
\end{aligned} \tag{5}
$$

On the other hand, it is well known that, for each g-orthogonal almost-complex structure I, I is integrable if and only if the covariant derivative $D^g I$ satisfies the following identity:

$$
D^g_X I + I \circ D^g_{IX} I = 0 \tag{6}
$$

for any vector field X. By (4) and (5), we get

$$
D^g I_1 + I_1 \circ D^g_{I_1 \cdot} I_1 = I_3\, (\theta_3 - \theta_2) \otimes I_2 + I_2\, (\theta_3 - \theta_2) \otimes I_3. \tag{7}
$$

It follows readily that I_1 is integrable if and only if $\theta_2 = \theta_3$ and similarly for I_2 and I_3. □

Note. The *only if* part of (a) is more or less obvious since $I_3 = I_1 \circ I_2$. The less obvious *if* part of (a) has been brought to the attention of the first author by F. Battaglia and S. Salamon.

In view of Proposition 2, the common Lee form of (g, I_1), (g, I_2) and (g, I_3) when I_1, I_2, I_3 are integrable will be called the *Lee form of the hyper-Hermitian structure*, denoted by θ.

The Lee form θ of a four-dimensional hyper-Hermitian structure satisfies $(\mathrm{d}\theta)_+ \equiv 0$, i.e. the exterior differential $\mathrm{d}\theta$ is anti-self-dual. This is because $W^+ \equiv 0$ by Proposition 1 [4]. In particular, if M is compact, $\mathrm{d}\theta \equiv 0$, i.e. any four-dimensional hyper-Hermitian structure is locally conformally hyper-Kähler, cf. [5] for a classification.

In the present paper, we focus on the opposite case, when M is not compact and hyper-Hermitian structures are *not* locally conformally hyper-Kähler.

2.2. *Distinguished hyper-Hermitian connections*

It is well known that an almost-hypercomplex structure $\{I_1, I_2, I_3\}$ is hypercomplex if and only if there exists a torsion-free linear connection preserving each I_i; then, such a connection is unique [3,13,17]. This holds in any dimension. Observe that the *if* part of the above assertion is obvious.

In dimension 4, the Obata connection D^{Ob} can be expressed as follows. Choose any Riemannian metric g in the conformal class determined by $\{I_1, I_2, I_3\}$ and, assuming

I_1, I_2, I_3 are integrable, denote by θ their common Lee form with respect to g as above. Then, via (4), it is easily checked that the linear connection D^{Ob} defined by

$$D^{\mathrm{Ob}}_X Y = D^g_X Y - \tfrac{1}{2}\theta(X)Y - \tfrac{1}{2}\theta(Y)X + \tfrac{1}{2}g(X, Y)\theta^\sharp \tag{8}$$

is independent of the choice of g in the conformal class and preserves each I_i, whence is equal to the Obata connection.

Moreover, D^{Ob} also preserves the conformal class $[g]$, as the latter is determined by the triple $\{I_1, I_2, I_3\}$, whence is a *Weyl connection* with respect to $[g]$.

Further, as shown in [15], D^{Ob} is in fact Einstein–Weyl.

Assuming that the I_i's are integrable and starting from the above defined Obata connection, it is an easy matter to construct hyper-Hermitian connections on TM with "small" torsion in some sense. As a matter of fact, we distinguish *two* "canonical" hyper-Hermitian connections, denoted by D^0 and D^1.

The connection D^0 is defined by

$$D^0 = D^{\mathrm{Ob}} + \tfrac{1}{2}\theta \otimes 1_{TM}, \tag{9}$$

where 1_{TM} denotes the identity of TM. Equivalently,

$$D^0_X = D^g_X - \theta \wedge X, \tag{10}$$

for any vector field X, where $\theta \wedge X$ stands for the (skew-symmetric) endomorphism of TM defined by $(\theta \wedge X)(Y) = \theta(Y)X - g(X, Y)\theta^\sharp$. Then, D^0 clearly preserves g and the I_i's, by (10) and (9) respectively. Moreover, its torsion T^{D^0} is "identified" with the 1-form θ as follows

$$T^{D^0}_{X,Y} = \tfrac{1}{2}(\theta(X)Y - \theta(Y)X). \tag{11}$$

The connection D^1 is defined by

$$D^1 = D^g - \tfrac{1}{2} * \theta, \tag{12}$$

where $*$ denotes the Hodge-star operator on M, i.e.

$$g(D^1_X Y, Z) = g(D^g_X Y, Z) - \tfrac{1}{2}(*\theta)(X, Y, Z). \tag{13}$$

By (12), the connection D^1 clearly preserves g. In order to check that D^1 also preserves the I_i's, we recall the following general identity:

$$*\alpha = I\alpha \wedge \Omega_I \tag{14}$$

for any 1-form α and *any* g-orthogonal almost-complex structure I, with Kähler form Ω_I. Applying the above identity to the Lee form θ, we get (same convention as for (10))

$$D^1_X = D^0_X + \tfrac{1}{2}\theta \wedge X + \tfrac{1}{2}I_i\theta \wedge I_i X + \tfrac{1}{2}\theta(I_i X)I_i \tag{15}$$

for *each* I_i, $i = 1, 2, 3$. It readily follows that D^1 preserves each I_i.

The torsion T^{D^1} of D^1 is *completely skew-symmetric*, identified with the 3-form $- * \theta$, as follows:

$$g(T^{D^1}_{X,Y}, Z) = -(*\theta)(X, Y, Z). \tag{16}$$

2.3. Tri-holomorphic Killing vector fields

We consider the case that M admits a non-trivial *tri-holomorphic Killing vector field* X, i.e. a non-trivial vector field X preserving the metric g and the almost-complex structures I_i's:

$$\mathcal{L}_X = 0, \qquad \mathcal{L}_X \mathcal{I}_i = 0, \tag{17}$$

for $i = 1, 2, 3$, where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X.

We denote by ξ the dual 1-form of X and by ξ_i the dual 1-form of the vector field $I_i\, X$ with respect to g, $i = 1, 2, 3$. The function $g(X, X)$ will be denoted by F.

Recall that X being a Killing vector field is equivalent to $D^g\xi$ being skew-symmetric, or, equivalently

$$D^g\xi = \tfrac{1}{2}\, \mathrm{d}\xi. \tag{18}$$

The expressions of $\mathrm{d}\xi = 2\, D^g\xi$ and of $\mathrm{d}\xi_i$, $i = 1, 2, 3$, are given by the following proposition where, for any vector field Y, $i_Y\, \mathrm{d}\xi$ stands for the inner product $\mathrm{d}\xi(Y, \cdot)$ of the 2-form $\mathrm{d}\xi$ by Y. Notice that, at any point where X is non-zero, any 2-form ϕ is entirely determined by the 1-forms $i_X\phi$ and $i_{I_i X}\phi$, $i = 1, 2, 3$. We then have:

Proposition 3.

(a) *The 2-form* $\mathrm{d}\xi$ *satisfies*

$$\begin{aligned} i_X\, \mathrm{d}\xi &= -\,\mathrm{d}F, \\ i_{I_1 X}\, \mathrm{d}\xi &= -I_1\, \mathrm{d}F + \alpha_2(I_2X)\, \xi_3 - \alpha_3(I_3X)\, \xi_2, \\ i_{I_2 X}\, \mathrm{d}\xi &= -I_2\, \mathrm{d}F + \alpha_3(I_3X)\xi_1 - \alpha_1(I_1X)\xi_3, \\ i_{I_3 X}\, \mathrm{d}\xi &= -I_3\, \mathrm{d}F + \alpha_1(I_1X)\xi_2 - \alpha_2(I_2X)\xi_1. \end{aligned} \tag{19}$$

where the 1-forms α_i *are defined by* (4).

In particular, X *is parallel if and only if its norm is constant and* $\alpha_i(I_iX)$ *vanishes identically for* $i = 1, 2, 3$.

(b) *The 2-forms* $\mathrm{d}\xi_i$, $i = 1, 2, 3$, *satisfy*

$$\mathrm{d}\xi_i = \theta_i \wedge \xi_i - \theta_i(X)\, \Omega_i. \tag{20}$$

Proof. Since X is Killing, we have $0 = \mathcal{L}_X\xi = \mathrm{d}(i_X\xi) + i_X\, \mathrm{d}\xi = -\mathrm{d}F + i_X\, \mathrm{d}\xi$. This gives the first equation in (19). In order to prove the three remaining equations, we start from the obvious identity $(D^g_X I_i)(X) = D^g_X(I_i\, X) - I_i(D^g_X X)$ for $i = 1, 2, 3$. Since X preserves I_i, we have $[X, I_iX] = 0$, and the above identity becomes $(D^g_X I_i)(X) = D^g_{I_iX} X - I_i(D^g_X X)$. Finally, because of (18), we get $(D^g_X I_i)(X) = \frac{1}{2} i_{I_iX}\, \mathrm{d}\xi - \frac{1}{2} I_i(i_X\, \mathrm{d}\xi)$. We then conclude by using (4).

The last assertion of (a) follows immediately.

Since X is Killing and preserves I_i, it also preserves the Kähler form Ω_i. We thus have $0 = \mathcal{L}_X\Omega_i = \mathrm{d}(i_X\Omega_i) + i_X(\mathrm{d}\Omega_i) = \mathrm{d}\xi_i + i_X(\theta_i \wedge \Omega_i) = \mathrm{d}\xi_i + \theta(X)\Omega_i - \theta_i \wedge \xi_i$, $i = 1, 2, 3$. This proves (20). □

2.4. The induced three-dimensional geometry

From now on, we assume that the vector field X is nowhere vanishing.

Then, M is locally fibred over some *Riemannian* 3-manifold Σ, defined as the manifold of trajectories of X in some open set of M. Restricting our attention to this open set, we shall assume that M itself is fibred over Σ.

Vector fields on Σ are identified with vector fields Y on M orthogonal to X at any point and satisfying $\mathcal{L}_X Y = 0$. In particular, the inner product of two such vector fields is constant along each trajectory of X and determines a well-defined Riemannian metric on Σ [11]. Differential forms on Σ are identified with differential forms ϕ on M satisfying $i_X\phi = 0$ and $\mathcal{L}_X\phi = 0$ or, equivalently, $i_X\phi = 0$ and $i_X\,\mathrm{d}\phi = 0$.

In particular, the vector fields I_iX, the 1-forms ξ_i and their exterior differential $\mathrm{d}\xi_i$ may be (and will be) considered as defined on Σ, for $i = 1, 2, 3$.

For further convenience, we introduce the positive function V defined by

$$V = \frac{1}{F}. \tag{21}$$

Then the metric g can be written as follows:

$$g = V\left(\xi \otimes \xi + \sum_{i=1}^{3} \xi_i \otimes \xi_i\right). \tag{22}$$

The function V being constant along the trajectories of X may and will be considered as defined on Σ.

The same is true for the 2-form $\mathrm{d}(V\xi)$ (but *not* for $\mathrm{d}\xi$ if V is not constant). Indeed, since X is Killing, we have $0 = \mathcal{L}_X\xi = i_X\,\mathrm{d}\xi + \mathrm{d}(i_X\xi) = V^{-1}\,i_X\,\mathrm{d}(V\,\xi)$.

We infer the existence of a (local) real function t on M, defined up to a function on Σ, such that

$$V\xi = \mathrm{d}t + \omega, \tag{23}$$

where ω is a 1-form on Σ, defined up to the differential of some function on Σ, satisfying $\mathrm{d}\omega = \mathrm{d}(V\,\xi)$. In particular, M is thus locally identified with the product $\mathbb{R} \times \Sigma$ in such a way that the Killing vector field X is identified with the vector field $\partial/\partial t$.

Instead of the metric induced by g on Σ, we shall consider the conformal metric g_Σ, defined by

$$g_\Sigma = \sum_{i=1}^{3} \xi_i \otimes \xi_i, \tag{24}$$

with respect to which the triple $\{\xi_i\}_{i=1,2,3}$ is a (direct) orthonormal (global) frame of the cotangent bundle $T^*\Sigma$. Then the dual orthonormal frame of the tangent bundle $T\Sigma$ is the triple $\{e_i = V\, I_i X\}_{i=1,2,3}$. Define a real 1-form $\tilde{\alpha}$ on Σ by

$$\tilde{\alpha} = \sum_{i=1}^{3} \alpha_i(e_i)\, \xi_i. \tag{25}$$

Then the system on M formed by the three last equations in (19) is equivalent to the following unique equation on Σ:

$$\mathrm{d}V + V\tilde{\alpha} = - *_\Sigma\, \mathrm{d}\omega, \tag{26}$$

where $*_\Sigma$ denotes the Hodge-star operator on Σ with respect to g_Σ and the induced orientation.

In the same way, it follows from (20) that, for $i = 1, 2, 3$, the function $\theta_i(X)$ and the 1-form $\theta_i^\perp$ defined by $\theta_i^\perp = \theta_i - V\,\theta_i(X)\,\xi$ are actually defined on Σ and that the system (20) on M is equivalent to the following system on Σ:

$$\mathrm{d}\xi_i = \theta_i^\perp \wedge \xi_i - V\,\theta_i(X)\, *_\Sigma\, \xi_i, \tag{27}$$

for $i = 1, 2, 3$.

If the almost-hyper-Hermitian structure $\{I_1, I_2, I_3\}$ is actually hypercomplex, the 1-form $\tilde{\alpha}$ coincides with $\theta^\perp$, where θ is the common Lee form θ of I_1, I_2, I_3 (cf. Propostion 2) and the system (26) and (27) on Σ reduces to

$$\mathrm{d}V + V\alpha = - *_\Sigma\, \mathrm{d}\omega, \tag{28}$$

$$\mathrm{d}\xi_i = \alpha \wedge \xi_i - \kappa *_\Sigma\, \xi_i \tag{29}$$

for $i = 1, 2, 3$, where α stands for $\theta^\perp = \theta - V\theta(X)\xi$ and κ and $V\theta(X)$.

The "converse" is true in the following sense:

Proposition 4. *Let (Σ, g_Σ) be a three-dimensional, oriented Riemannian manifold. Let $\{\xi_i\}_{i=1,2,3}$ be a direct, orthonormal coframe of (Σ, g_Σ) satisfying* (29) *for some real 1-form α and some real function κ. Denote by $\{e_i\}_{i=1,2,3}$ the dual orthonormal frame of $T\Sigma$.*

Let V be a positive real function and ω a real 1-form satisfying (28) *for the same 1-form α. Denote by M the product $\mathbb{R} \times \Sigma$ and by t the canonical parameter of the factor $\mathbb{R}$. Consider the vector field X and the real 1-form ξ on M defined by $X = \partial/\partial t$ and $\xi = V^{-1}(\mathrm{d}t + \omega)$.*

Let g be the bilinear form on M defined by $g \doteq V(\xi \otimes \xi + g_\Sigma)$.

Let $\{I_i\}_{i=1,2,3}$ be the triple of almost-complex structures on M determined by $I_1 X = V^{-1}e_1$, $I_1 e_2 = e_3$; $I_2 X = V^{-1}e_2$, $I_2 e_3 = e_1$; $I_3 X = V^{-1}e_3$, $I_3 e_1 = e_2$.

Then

(a) *g is positive-definite on $M = \mathbb{R} \times \Sigma$ and X and ξ are dual of each other with respect to g.*

(b) I_1, I_2, I_3 *are integrable and, together with* g, *form a hyper-Hermitian structure on* M, *whose Lee form is equal to*

$$\theta = \alpha + \kappa\,\xi = \alpha + \frac{\kappa}{V}(\mathrm{d}t + \omega). \tag{30}$$

(c) *The vectorfield* X *is a tri-holomorphic Killing vectorfield with respect to* $\{g, I_1, I_2, I_3\}$.

Proof. (a) We get a nowhere vanishing vector field T g-orthogonal to Σ by putting $T = \partial/\partial t - \omega^\sharp/(|\omega|^2 + V^2)$, where $\omega^\sharp$ denotes the dual vector field of ω with respect to g_Σ on Σ. An easy computation shows that $g(T, T) = V/(|\omega|^2 + V^2)$, which is everywhere positive. This proves that g is everywhere positive-definite. The remaining assertion is obvious. (b) The almost-complex structures I_i are clearly g-orthogonal and satisfy (1). The Kähler form Ω_1 is equal to $\Omega_1 = V(\xi \wedge \xi_1 + \xi_2 \wedge \xi_3)$. By computing $\mathrm{d}\Omega_1$, via (28) and (29), and using (3), we easily infer that the corresponding Lee form θ_1 is equal to $\alpha + \kappa\,\xi$. The same is true for I_2 and I_3. It follows that I_1, I_2 and I_3 have the same Lee form with respect to g, whence are integrable by Propostion (c). Since the expressions of g and I_i, $i = 1, 2, 3$, do not explicitly involve t, all are preserved by $X = \partial/\partial t$. □

The problem of determining the hyper-Hermitian metric (22) has now fallen into two parts: first, one must determine the three-dimensional geometry on Σ which satisfies (29); then, one must solve the monopole-like equation (28) for the pair (V, ω).

One knows from general theory [11] that the space of orbits Σ of an isometry (even merely a conformal isometry) in a Riemannian 4-manifold M with $W^+ = 0$ will admit an Einstein–Weyl structure, with metric defined as in (24). Thus one anticipates that (29) will define an Einstein–Weyl geometry, while (28) is a covariant equation in this geometry. The conformal freedom

$$g_\Sigma \mapsto f^2 g_\Sigma,\quad \xi_i \mapsto f\,\xi_i,\quad \alpha \mapsto \alpha + \frac{\mathrm{d}f}{f},\quad \kappa \mapsto f^{-1}\kappa,\quad V \mapsto f^{-1}V, \tag{31}$$

for any positive function f on Σ, is easily seen to be an invariance of (28) and (29).

3. Einstein–Weyl geometry

In this section, we first investigate the system (29) for a coframe $\{\xi_i\}_{i=1,2,3}$ which is orthogonal with respect to the Riemannian metric g_Σ. As before, we denote by $\{e_i\}_{i=1,2,3}$ the dual g_Σ-orthonormal frame of $T\Sigma$.

Since $\{\xi_i\}$ is orthonormal, it is easy to check that the system (29) is equivalent to the following system:

$$\begin{aligned}
D^{g_\Sigma}\xi_1 &= -\xi_1 \otimes \alpha + \alpha(e_1)\,g_\Sigma - \tfrac{1}{2}\kappa\,\xi_2 \wedge \xi_3,\\
D^{g_\Sigma}\xi_2 &= -\xi_2 \otimes \alpha + \alpha(e_2)\,g_\Sigma - \tfrac{1}{2}\kappa\,\xi_3 \wedge \xi_1,\\
D^{g_\Sigma}\xi_3 &= -\xi_3 \otimes \alpha + \alpha(e_3)\,g_\Sigma - \tfrac{1}{2}\kappa\,\xi_1 \wedge \xi_2,
\end{aligned} \tag{32}$$

where D^{g_Σ} denotes the Levi-Civita connection of g_Σ (here $D^{g_\Sigma}\xi_1$, etc. is viewed as a real bilinear form).

Let ∇^0 denote the uniquely defined linear connection on the tangent bundle $T\Sigma$ satisfying

$$\nabla^0 e_i = 0, \quad i = 1, 2, 3. \tag{33}$$

By its very definition, the connection ∇^0 is flat and preserves the metric g (but has torsion).

Then, the system (32) is *equivalent* to the fact that the connection ∇^0 determined by the (unknown) orthonormal frame $\{e_i\}_{i=1,2,3}$ is related to D^{g_Σ} by

$$\nabla^0_Y Z = D^{g_\Sigma}_Y Z + g_\Sigma(Y, Z)\,\alpha^\sharp - \alpha(Z)\,Y - \tfrac{1}{2}\kappa\,(i_Z(*_\Sigma Y^\flat))^\sharp \tag{34}$$

for any vector fields Y, Z on Σ, where the duality isomorphisms $\sharp$ and $\flat$ are relative to the metric g_Σ.

The point is that the connection ∇^0 determined by the *unknown* orthonormal frame $\{e_i\}_{i=1,2,3}$ is now well-defined by (34). Hence, a necessary and sufficient condition for the system (29) to admit a local solution $\{\xi_i\}_{i=1,2,3}$ is that the connection ∇^0 *defined* by (34), which obviously preserves the metric g_Σ, is flat. Indeed, ∇^0 is flat if and only if $T\Sigma$ can be locally trivialized by a ∇^0-parallel, g_Σ-orthonormal frame $\{e_i\}_{i=1,2,3}$ and any such frame satisfies (32), hence (29) .

The problem can be re-formulated by introducing the *Weyl-connection* D defined by

$$D_Y Z = D^{g_\Sigma}_Y Z - \alpha(Y)Z - \alpha(Z)Y + g_\Sigma(Y, Z)\alpha^\sharp \tag{35}$$

for any vector fields Y, Z.

Then (34) can also be written in the following way:

$$\nabla^0_Y Z = D_Y Z + \alpha(Y)Z - \tfrac{1}{2}\kappa\,(i_Z(*_\Sigma Y^\flat))^\sharp. \tag{36}$$

or, more briefly

$$\nabla^0_Y = D_Y + \alpha(Y)\,1_{T\Sigma} - \tfrac{1}{2}\kappa\, *_\Sigma Y^\flat, \tag{37}$$

where $1_{T\Sigma}$ denotes the identity of $T\Sigma$ and $*_\Sigma Y^\flat$ is here considered as a skew-symmetric endomorphism of $T\Sigma$ via the metric g_Σ.

Then the curvature R^0 of ∇^0 is related to the curvature R^D of the Weyl connection D by

$$\begin{aligned} R^0_{X,Y} &= R^D_{X,Y} - \mathrm{d}\alpha(X, Y)1_{T\Sigma} - \tfrac{1}{4}\kappa^2 X \wedge Y \\ &\quad + \tfrac{1}{2}(\mathrm{d}\kappa + \kappa\alpha)(X) *_\Sigma Y - \tfrac{1}{2}(\mathrm{d}\kappa + \kappa\alpha)(Y) *_\Sigma X \\ &= (\tfrac{1}{6}\mathrm{Scal}^D_g - \tfrac{1}{4}\kappa^2)\, X \wedge Y + \mathrm{Ric}^D_0(X) \wedge Y + X \wedge \mathrm{Ric}^D_0(Y) \\ &\quad + \tfrac{1}{2}(\mathrm{d}\kappa + \kappa\alpha - *\,\mathrm{d}\alpha)(X) *_\Sigma Y - \tfrac{1}{2}(\mathrm{d}\kappa + \kappa\alpha - *\,\mathrm{d}\alpha)(Y) *_\Sigma X, \end{aligned} \tag{38}$$

where $\mathrm{Scal}^D_{g_\Sigma}$ denotes the scalar curvature of D with respect to g_Σ and Ric^D_0 denotes the symmetric, trace-free part of the Ricci tensor of D (here, R^0 and R^D are considered as endomorphisms of $\Lambda^2(T\Sigma)$).

It is easily checked that the first line in the second part of the RHS of (38) always satisfies the (first) Bianchi identity, while the last line satisfies the Bianchi identity if and only if it vanishes identically.

We thus finally get the following proposition.

Proposition 5. *Let* (Σ, g_Σ) *be a connected, three-dimensional Riemannian manifold. Then the system* (29) *is locally solvable, for an orthonormal coframe* $\{\xi_i\}_{i=1,2,3}$ *if and only if the connection* ∇^0 *defined by* (34) *is flat. This happens if and only if the curvature* R^D *of the Weyl connection D defined by* (35) *satisfies the three following conditions:*

(S) $\mathrm{Scal}^D_{g_\Sigma} = \frac{3}{2}\kappa^2$,

(E) $\mathrm{Ric}^D_0 \equiv 0$,

(K) $\mathrm{d}\kappa + \kappa\alpha - *_\Sigma \,\mathrm{d}\alpha = 0$.

Moreover, any solution $\{\xi_i\}_{i=1,2,3}$ *is uniquely determined by its value at some point of M, which can be chosen arbitrarily, and can be extended to a global solution whenever M is simply connected.*

It is easily observed that the whole problem is invariant under the *gauge transformation* (31). In particular, the latter preserves the Weyl connection D and the three conditions (S)–(E)–(K).

Condition (E) means that D is an *Einstein–Weyl connection.*

Condition (S) expresses the fact that *the (conformal) scalar curvature* Scal^D *of D is a square*, i.e. Scal^D admits a square root, a section of the real line bundle L^{-1} (cf. Remark 1) whose expression with respect to the metric g_Σ is here denoted by κ, up to a factor $\sqrt{3/2}$. In particular, for any metric g_Σ in the conformal class, the real function $\mathrm{Scal}^D_{g_\Sigma}$ is non-negative everywhere.

Condition (K) is a conformally invariant additional condition for the square-root of Scal^D, whose meaning is explained in Remark 1.

Since conditions (S)–(E)–(K) are "gauge invariant", Proposition 5 can be re-formulated as follows:

Proposition 6. *Let* $(\Sigma, [g_\Sigma])$ *be a connected, three-dimensional, conformal manifold. Let D be a Weyl connection with respect to* $[g_\Sigma]$ *whose scalar curvature is a square. For any metric* g_Σ *in the conformal class* $[g_\Sigma]$, *let* α *be the real 1-form defined by* (35) *and* κ *a real function satisfying* (S).

Then, for any metric g_Σ *in* $[g_\Sigma]$, *the system* (29) *is locally solvable for a* g_Σ*-orthonormal coframe* $\{\xi_i\}_{i=1,2,3}$ *if and only if D satisfies the two conditions* (E) *and* (K).

If, moreover, Σ *is compact, if D satisfies* (E)–(K) *and if* g_Σ *is the standard metric determined by D, so that* α *is the dual of a Killing vector field, then* κ *is constant, as are the Riemannian scalar curvature* Scal^{g_Σ} *and the norm of* α. *It then follows that the Einstein–Weyl manifold* (Σ, g_Σ, D) *is either Einstein (in the Riemannian sense) or, up to a finite covering, isomorphic to* $S^1 \times S^2$, *equipped with one of its two canonical flat Weyl structures or a Berger sphere with its canonical Einstein–Weyl structure, according as* κ *is zero or positive.*

Proof. The first part is a re-formulation of Proposition 5.

If Σ is compact, for a given Einstein–Weyl connection D, we can choose for g_Σ in $[g_\Sigma]$ the *standard metric* of D, so that the 1-form α is co-closed, i.e. $*_\Sigma\, \mathrm{d}\alpha$ is closed. From (c), we then infer that κ satisfies: $\Delta_\Sigma \kappa + g_\Sigma(\mathrm{d}\kappa, \alpha) = 0$, hence is *constant*. By [9, I (31)–(32)], it then follows that the Riemannian scalar curvature Scal^{g_Σ} and the square-norm $|\alpha|^2$ of α are constant as well. The last statement is a direct consequence of the classification in [18]. □

Remark 1. When no metric is chosen in the conformal class $[g_\Sigma]$, the scalar curvature Scal^D of D has to be considered as a section of the real line bundle L^{-2} of scalars of weight -2; then, κ appears, by (S), as a section of L^{-1}. For any integer k, we denote by $\nabla^{(k)}$ the connection induced by D on L^k, by $\rho^{(k)}$ its curvature, equal to the real 2-form $k\,\mathrm{d}\alpha$. We denote by $\mathrm{d}^{\nabla^{(k)}}$ the exterior differential with respect to $\nabla^{(k)}$, so that $(\mathrm{d}^{\nabla^{(k)}} \circ \nabla^{(k)}) = -k\,\mathrm{d}\alpha$. Then, for any section κ of L^{-1}, (K) reads as follows:

$$\nabla^{(-1)}\kappa = - *_\Sigma\, \rho^{(-1)}, \tag{39}$$

a conformally invariant monopole equation.

We recall the (conformal) Bianchi identity, which, for an Einstein–Weyl structure, reads as follows [9, Lemma 2; 16]:

$$\tfrac{1}{3} *_\Sigma \nabla^{(-2)}(\mathrm{Scal}^D) + \mathrm{d}^{\nabla^{(-1)}}(*_\Sigma \rho^{(-1)}) = 0. \tag{40}$$

For any section κ of L^{-1}, we consider the quantity $Q = \nabla^{(-1)}\kappa + *_\Sigma \rho^{(-1)}$, an L^{-1}-valued 1-form on Σ. By differentiating Q and using (40) and (S), we easily get the identity

$$\mathrm{d}^{\nabla^{(-1)}} Q + \kappa \otimes *_\Sigma Q = \tfrac{1}{3} *_\Sigma \nabla^{(-2)}(\mathrm{Scal}^D) + \mathrm{d}^{\nabla^{(-2)}}(*_\Sigma \rho^{(-1)}) = 0. \tag{41}$$

Thus, we may think of the Bianchi identity as a linear equation for Q. The content of (39) is that Q is the solution zero of this equation and that therefore we cannot obtain more conditions by differentiating (39).

To find an explicit example satisfying (K), one can check directly that the canonical Einstein–Weyl structure on the Berger sphere [11] does so as is implicit in Proposition 6. Conversely, as shown in Proposition 6, a compact Einstein–Weyl manifold satisfying (K) is either locally conformally Einstein or is the Berger sphere. Without imposing compactness, one may perform a local calculation to find that Einstein–Weyl spaces subject to (K) are determined by two free functions of two variables, whereas the general three-dimensional Einstein–Weyl space is determined by four such functions [6] .

There is a class of three-dimensional Einstein–Weyl spaces determined by a solution of the so-called $SU(\infty)$ *Toda field equation* [19] and depending on two free functions of two variables (such Einstein–Weyl structures appear in constructions by C. LeBrun of complex Kähler surfaces with vanishing scalar curvature and self-dual metrics on connected sums of complex projective planes [12]). If we impose (K) on these solutions, we find that the only possibilities are locally conformally Einstein. Thus, the new class of solutions is complementary to these previously known ones.

We may investigate the monopole equation (28) for a three-dimensional Riemannian manifold (Σ, g_Σ) admitting a Weyl-connection D satisfying the conditions (S)–(E)–(K) of Proposition 5, defined by (35). The 1-form α appearing in (35), which depends upon the choice of the metric g_Σ in the conformal class $[g_\Sigma]$, will be called *the potential 1-form of the Weyl connection D with respect to the metric g_Σ*. Then (28) can be interpreted in the following way:

Proposition 7. *The positive function V is a local solution of* (28) *for some (local) 1-form ω on Σ if and only if the potential 1-form $\tilde{\alpha}$ of the Weyl connection D with respect to the conformal metric $\tilde{g}_\Sigma = V^2 g_\Sigma$ is $\tilde{g}_\Sigma$-coclosed.*

Proof. The monopole equation (28) is locally solvable in ω if and only if the 2-form $*_\Sigma(\mathrm{d}V + V\alpha)$ is closed. The Hodge operator $\tilde{*}_\Sigma$ of the metric $\tilde{g}_\Sigma$, operating on 1-forms, is equal to $V *_{g_\Sigma}$, while $\tilde{\alpha}$ is related to α by $\tilde{\alpha} = \alpha + \mathrm{d}V/V$. We thus have $\tilde{*}_\Sigma\tilde{\alpha} = *_\Sigma(V\alpha + \mathrm{d}V)$. The conclusion follows immediately. □

As a conclusion, via Propositions 4 and 6, we obtain four-dimensional hyper-Hermitian structures by observing the following prescriptions:

(1) Consider a three-dimensional Einstein–Weyl space $(\Sigma, [g_\Sigma], D)$ satisfying the additional condition (K) of Proposition 5 for some section κ of L^{-1} related to the scalar curvature of D by (S).
(2) For any metric g_Σ in the conformal class, choose a positive function V (if any) so that the potential of D with respect to $V^2 g_\Sigma$ is $V^2 g_\Sigma$-coclosed (i.e. the metric $V^2 g_\Sigma$ is *standard* with respect to D [8]) and consider some (local) 1-form ω on Σ such that (28) is satisfied.
(3) Apply the recipe of Proposition 4 to obtain a hyper-Hermitian structure on the product $M = \mathbb{R} \times \Sigma$.

The examples given by Papadopoulos [14] are based on the Einstein–Weyl space which is locally conformal to the Einstein metric on S^3.

Remark 2. A case when the above mentioned prescription (2) can be trivially satisfied is the case that $\mathrm{Scal}^{\mathrm{D}}$ is strictly positive (with respect to any metric), so that κ can be chosen positive (up to a change of the orientation of Σ). We then obtain a solution of (28) by putting $V = \kappa$, $\omega = -\alpha$. By (30), the Lee form of the corresponding hyperhermitian structure is then equal to $\mathrm{d}t$; in particular, the hyper-Hermitian structure is then locally conformally hyper-Kähler.

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References

[1] V. Apostolov and P. Gauduchon, A Riemannian Goldberg–Sachs Theorem, Internat. J. Math., to appear.

[2] M.F. Atiyah, N.J. Hitchin and I.M. Singer, Self-duality in four dimensional geometry, Proc. Roy. Soc. London A 362 (1979) 425–461.

[3] E. Bonan, Sur les G-structures de type quaternionien, Cahiers de Topologie et Géométrie Différentielle, Vol. 9 (1967) pp. 389–461.

[4] C. Boyer, Conformal duality and compact complex surfaces, Math. Ann. 274 (1986) 517–526.

[5] C. Boyer, A note on hyperhermitian four-manifolds, Proc. Amer. Math. Soc. 102 (1988) 157–164.

[6] E. Cartan, Sur une classe d'espaces de Weyl, Ann. Sci. Ec. Norm. Sup. 60 (1943) 1–16.

[7] T. Chave, K.P. Tod and G. Valent (4, 0) and (4, 4) sigma models with a triholomorphic Killing vector, Phys. Lett. B 383 (1996) 262–270.

[8] P. Gauduchon, La 1-forme de torsion d'une variété hermitienne compacte, Math. Ann. 267 (1984) 495–518.

[9] P. Gauduchon, Structures de Weyl–Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$, J. Reine Angew. Math. 469 (1995) 1–50.

[10] N.J. Hitchin, On compact four dimensional Einstein manifolds, J. Differential Geom. 9 (1974) 435–442.

[11] P.E. Jones and K.P. Tod, Minitwistor spaces and Einstein–Weyl spaces, Class. Quant. Grav. 2 (1985) 565–577.

[12] C. LeBrun, Explicit self-dual metrics on $\mathbb{C}P^2\# \cdots \#\mathbb{C}P^2$, J. Differential Geom. 34 (1991) 233–253.

[13] M. Obata, Affine connections on manifolds with almost complex quaternionic or Hermitian structures, Jap. J. Math. 26 (1956) 43–79.

[14] G. Papadopoulos, Elliptic monopoles and (4, 0)-supersymmetric sigma models with torsion, Phys. Lett. B 356 (1995) 249–255.

[15] H. Pedersen and A. Swann, Riemannian submersions, four-manifolds and Einstein geometry, Proc. London Math. Soc. 3 (66) (1993) 381–399.

[16] H. Pedersen and K.P. Tod, Three-dimensional Einstein–Weyl geometry, Adv. Math. 97 (1993) 74–109.

[17] S. Salamon, *Riemannian Geometry and Holonomy Groups*, Pitman Research Notes in Math. Series, Vol. 201 (Longman, New York, 1989).

[18] K.P. Tod, Compact three-dimensional Einstein–Weyl structures, J. London Math. Soc. (2) 45 (1992) 341–351.

[19] R.S. Ward, Einstein–Weyl spaces and $SU(\infty)$ Toda fields, Class. Quant. Grav. 7 (1990) L95–L98.